

# Resonant Tidal Excitations of Inertial Modes in Coalescing Neutron Star Binaries

Dong Lai\*

*Center for Radiophysics and Space Research, Department of Astronomy, Cornell University, Ithaca, NY 14853*

Yanqin Wu

*Department of Astronomy and Astrophysics, University of Toronto,  
60 St. George Street, Toronto, ON M5S 3H8, Canada*

We study the effect of resonant tidal excitation of inertial modes in neutron stars during binary inspiral. For spin frequencies less than 100 Hz, the phase shift in the gravitational waveform associated with the resonance is small and does not affect the matched filtering scheme for gravitational wave detection. For higher spin frequencies, the phase shift can become significant. Most of the resonances take place at orbital frequencies comparable to the spin frequency, and thus significant phase shift may occur only in the high-frequency band (hundreds of Hertz) of gravitational wave. The exception is a single odd-parity  $m = 1$  mode, which can be resonantly excited for misaligned spin-orbit inclinations, and may occur in the low-frequency band (tens of Hertz) of gravitational wave and induce significant ( $\gg 1$  radian) phase shift.

## I. INTRODUCTION

Coalescing neutron star-neutron star (NS-NS) and neutron star-black hole (NS-BH) binaries are the most promising sources of gravitational waves (GWs) for ground-based detectors such as LIGO and VIRGO[1]. The last few minutes of the binary inspiral produce GWs with frequencies sweeping upward through the 10–1000 Hz range (LIGO’s sensitivity band)[2]. Due to the expected low signal-to-noise ratios, accurate gravitational waveforms are required to serve as theoretical templates that can be used to detect the GW signal from the noise and to extract binary parameters from the waveform.

In the early stage of the inspiral, with the GW frequencies between 10 Hz to a few hundred Hz, it is usually thought that the NS can be treated as a point mass, and tidal effects are completely negligible. This is indeed the case for the “quasi-equilibrium” tides as the tidal interaction potential scales as  $a^{-6}$  (where  $a$  is the orbital separation)[3]. The situation is more complicated for the *resonant tides*: As two compact objects inspiral, the orbit can momentarily come into resonance with the normal oscillation modes of the NS. By drawing energy from the orbital motion and resonantly exciting the modes, the rate of inspiral is modified, giving rise to a phase shift in the gravitational waveform. This problem was first studied [4, 5] in the case of non-rotating NSs where the only modes that can be resonantly excited are g-modes (with typical mode frequencies  $\lesssim 100$  Hz). It was found that the effect is small for typical NS parameters (mass  $M \simeq 1.4M_\odot$  and radius  $R \simeq 10$  km) because the coupling between the g-mode and the tidal potential is weak. Ho & Lai (1999)[6] studied the effect of NS rotation, and found that the g-mode resonance can be strongly enhanced even

by a modest rotation (e.g., the phase shift in the waveform  $\Delta\Phi$  reaches up to 0.1 radian for a spin frequency  $\nu_s \lesssim 100$  Hz). They also found that for a rapidly rotating NS ( $\nu_s \gtrsim 500$  Hz), f-mode resonance becomes possible (since the inertial-frame f-mode frequency can be significantly reduced by rotation) and produces a large phase shift. In addition, NS rotation gives rise to r-mode resonance whose effect is appreciable only for very rapid (near breakup) rotations. Recently, Flanagan & Racine (2006)[7] studied the gravitomagnetic resonant excitation of r-modes and found that the post-Newtonian effect is more important than the Newtonian tidal effect (and the phase shift reaches  $\sim 0.1$  radian for  $\nu_s \sim 100$  Hz). Taken together, these previous studies suggest that for astrophysically most likely NS parameters ( $M \simeq 1.4M_\odot$ ,  $R \simeq 10$  km,  $\nu_s \lesssim 100$  Hz), tidal resonances have a small effect on the gravitational waveform during binary inspiral.

A rotating NS also supports a large number of Coriolis-force driven modes named inertial modes (also called rotational hybrid modes or generalized r-modes; see, e.g., Refs. [8, 9, 10, 11]), of which r-mode is a member. The modes have frequencies of order the spin frequency, and as we show below, they couple more strongly to the (Newtonian) tidal potential than the r-mode. It is therefore important to investigate the effect of inertial mode resonances on the GW phase evolution during binary inspiral — this is the goal of this paper.

## II. BASIC EQUATIONS FOR TIDAL RESONANCE DURING BINARY INSPIRAL

We consider a NS of mass  $M$ , radius  $R$  and spin  $\Omega_s$  in orbit with a companion  $M'$  (another NS or a black hole). The orbital radius  $a$  decreases in time due to GW emission,  $\Omega_{\text{orb}}$  is the orbital angular frequency. We allow for a general spin-orbit inclination angle  $\Theta$  (the angle between  $\Omega_s$  and the orbital angular momentum  $\mathbf{L}$ ). In the spherical coordinate system centered on  $M$  with the

---

\*Electronic address: dong@astro.cornell.edu

Z-axis along  $\mathbf{L}$ , the gravitational potential produced by  $M'$  can be expanded in terms of spherical harmonics:

$$U(\mathbf{r}, t) = -GM' \sum_{lm'} \frac{W_{lm'} r^l}{a^{l+1}} e^{-im' \Phi(t)} Y_{lm'}(\theta_L, \phi_L), \quad (1)$$

where  $\Phi(t) = \int^t dt \Omega_{\text{orb}}$  is the orbital phase and [12]

$$W_{lm'} = (-)^{(l+m')/2} \left[ \frac{4\pi}{2l+1} (l+m')! (l-m')! \right]^{1/2} \times \left[ 2^l \left( \frac{l+m'}{2} \right)! \left( \frac{l-m'}{2} \right)! \right]^{-1}, \quad (2)$$

[Here the symbol  $(-)^p$  is zero if  $p$  is not an integer.] Only the  $l \geq 2$  terms are relevant, and the dominant ( $l = 2$ ) tidal potential has  $W_{2\pm 2} = (3\pi/10)^{1/2}$ . Since it is most convenient to describe oscillation modes relative to the spin axis, we need to express the tidal potential in terms of  $Y_{lm}(\theta, \phi)$ , the spherical harmonic function defined in the corotating frame of the NS with the z-axis along  $\Omega_s$ . This is achieved by the relation

$$Y_{lm'}(\theta_L, \phi_L) = \sum_m \mathcal{D}_{mm'}^{(l)}(\Theta) Y_{lm}(\theta, \phi_s), \quad (3)$$

where the  $\mathcal{D}_{mm'}^{(l)}$  is the Wigner  $\mathcal{D}$ -function (e.g., Wybourne 1974), and  $\phi_s = \phi + \Omega_s t$ .

The linear perturbation of the tidal potential on the NS is specified by the Lagrangian displacement,  $\xi(\mathbf{r}, t)$ , of a fluid element from its unperturbed position. In the rotating frame of the NS, the equation of motion takes the form

$$\frac{\partial^2 \xi}{\partial t^2} + 2\Omega_s \times \frac{\partial \xi}{\partial t} + \mathbf{C} \cdot \xi = -\nabla U, \quad (4)$$

where  $\mathbf{C}$  is a self-adjoint operator (a function of the pressure and gravity perturbations) acting on  $\xi$  (see, e.g., Ref.[13]). A free mode of frequency  $\omega_\alpha$  with  $\xi_\alpha(\mathbf{r}, t) = \xi_\alpha(\mathbf{r}) e^{-i\omega_\alpha t} \propto e^{im\phi - i\omega_\alpha t}$  satisfies

$$-\omega_\alpha^2 \xi_\alpha - 2i\omega_\alpha \Omega_s \times \xi_\alpha + \mathbf{C} \cdot \xi_\alpha = 0, \quad (5)$$

where  $\{\alpha\}$  denotes the mode index, which includes the azimuthal “quantum” number  $m$ . We carry out phase space mode expansion [10, 14]

$$\left[ \frac{\partial \xi}{\partial t} \right] = \sum_\alpha c_\alpha(t) \left[ \begin{array}{c} \xi_\alpha(\mathbf{r}) \\ -i\omega_\alpha \xi_\alpha(\mathbf{r}) \end{array} \right]. \quad (6)$$

Using the orthogonality relation [13]  $\langle \xi_\alpha, 2i\Omega_s \times \xi_{\alpha'} \rangle + (\omega_\alpha + \omega_{\alpha'}) \langle \xi_\alpha, \xi_{\alpha'} \rangle = 0$  (for  $\alpha \neq \alpha'$ ), where  $\langle A, B \rangle \equiv \int d^3x \rho (A^* \cdot B)$ , we find [10]

$$\begin{aligned} \dot{c}_\alpha + i\omega_\alpha c_\alpha &= \frac{i}{2\varepsilon_\alpha} \langle \xi_\alpha(\mathbf{r}), -\nabla U \rangle \\ &= \sum_{m'} f_{\alpha, m'} e^{im\Omega_s t - im' \Phi}, \end{aligned} \quad (7)$$

with

$$f_{\alpha, m'} = \frac{iGM'}{2\varepsilon_\alpha} \sum_l \frac{W_{lm'}}{a^{l+1}} \mathcal{D}_{mm'}^{(l)} Q_{\alpha, lm}, \quad (8)$$

where

$$Q_{\alpha, lm} \equiv \langle \xi_\alpha, \nabla(r^l Y_{lm}) \rangle, \quad (9)$$

$$\varepsilon_\alpha \equiv \omega_\alpha + \langle \xi_\alpha, i\Omega_s \times \xi_\alpha \rangle, \quad (10)$$

and we have used the normalization  $\langle \xi_\alpha, \xi_\alpha \rangle = 1$ . Recall that in Eqs. (7-9), the index  $\alpha$  includes  $m$ .

Now consider the excitation of a specific mode with inertial-frame frequency  $\sigma_\alpha = \omega_\alpha + m\Omega_s$  by the potential component  $\propto e^{-im' \Phi}$  — we call this  $(\alpha, m')$ -resonance. In the following we shall adopt the convention  $m > 0$  and  $m' > 0$ . We first consider the case of  $\sigma_\alpha > 0$  (i.e., the mode is prograde with respect to the spin in the inertial frame, although in the rotating frame the mode can be either prograde or retrograde, corresponding to  $\omega_\alpha > 0$  and  $\omega_\alpha < 0$  respectively). The resonance occurs when

$$\sigma_\alpha = m' \Omega_{\text{orb}}. \quad (11)$$

Note that the mode azimuthal index  $m = m'$  for aligned spin-orbit ( $\Theta = 0$ ), but in general the  $m'$ -th potential can excite a mode with a different  $m$ . Integrating Eq. (7) across the resonance, we find that the post-resonance amplitude is given by

$$c_\alpha e^{i\omega_\alpha t} = \int dt f_{\alpha, m'} e^{i\sigma_\alpha t - im' \Phi} \simeq f_{\alpha, m'} \left( \frac{2\pi}{m' \dot{\Omega}_{\text{orb}}} \right)^{1/2}, \quad (12)$$

where  $a, \Omega_{\text{orb}}$  should be evaluated at the resonance radius  $a_\alpha$ , at which  $\sigma_\alpha = m' [G(M + M')/a_\alpha^3]^{1/2}$ . The mode energy in the inertial frame is  $2\sigma_\alpha \varepsilon_\alpha |c_\alpha|^2$ , where the factor 2 accounts for the fact that for each  $m > 0$ ,  $\sigma_\alpha > 0$  resonant mode (recall that our convention is  $m > 0$ ), there is also an identical  $m < 0$ ,  $\sigma_\alpha < 0$  resonant mode, and they should be counted as the same mode. Thus the energy transfer to the mode during the resonance is given by

$$\begin{aligned} \Delta E_{\alpha, m'} &= \frac{GM'^2}{R} \frac{GM}{R^3} \left( \frac{\pi}{m' \dot{\Omega}_{\text{orb}}} \right) \frac{\sigma_\alpha}{\varepsilon_\alpha} \\ &\times \left[ \sum_l W_{lm'} \mathcal{D}_{mm'}^{(l)} Q_{\alpha, lm} \left( \frac{R}{a_\alpha} \right)^{l+1} \right]^2. \end{aligned} \quad (13)$$

where we have defined  $Q_{\alpha, lm}$  in units such that  $M = R = 1$ . Except for differences in notation and sign convention, Eq. (13) agrees with the expression derived in Ref. [6], where mode decomposition was not carried out rigorously [15, 16].

The phase shift in the gravitational waveform due to the resonant energy transfer is twice the orbital phase shift and is given by

$$\Delta \Phi_{\text{GW}} = -2\Omega_{\text{orb}} t_{\text{GW}} \frac{\Delta E_{\alpha, m'}}{|E_{\text{orb}}|}, \quad (14)$$

where  $E_{\text{orb}} = -GMM'/(2a)$  is the orbital energy, and

$$t_{\text{GW}} = |a/\dot{a}| = \frac{5c^5 a^4}{G^3 M^3 q(1+q)} \quad (15)$$

is the orbital decay timescale ( $q = M'/M$  is the mass ratio; all quantities should be evaluated at the resonance radius  $a_\alpha$ ). Keeping the leading  $l$ -term in Eq. (13), we find

$$\Delta\Phi_{\text{GW}} = -\frac{25\pi}{1536} \left(\frac{Rc^2}{GM}\right)^5 \frac{1}{q(1+q)^{(2l-1)/3}} \times \frac{1}{\hat{\varepsilon}_\alpha} \left(\frac{\hat{\sigma}_\alpha}{m'}\right)^{(4l-11)/3} \left(W_{lm'} \mathcal{D}_{mm'}^{(l)} Q_{\alpha,lm}\right)^2, \quad (16)$$

where  $\hat{\sigma}_\alpha = \sigma_\alpha(R^3/GM)^{1/2}$  and  $\hat{\varepsilon}_\alpha = \varepsilon_\alpha(R^3/GM)^{1/2}$ .

For modes with  $\sigma_\alpha < 0$  (i.e., retrograde with respect to spin in the inertial frame), the GW phase shift at the resonance  $-\sigma_\alpha = m'\Omega_{\text{orb}}$  can be similarly derived and is given by

$$\Delta\Phi_{\text{GW}} = \frac{25\pi}{1536} \left(\frac{Rc^2}{GM}\right)^5 \frac{1}{q(1+q)^{(2l-1)/3}} \times \frac{1}{\hat{\varepsilon}_\alpha} \left(\frac{|\hat{\sigma}_\alpha|}{m'}\right)^{(4l-11)/3} \left(W_{l,-m'} \mathcal{D}_{m,-m'}^{(l)} Q_{\alpha,lm}\right)^2 \quad (17)$$

### III. INERTIAL MODES AND TIDAL COUPLING COEFFICIENTS

To fix notations, we first summarize the basic property of inertial modes for incompressible stars[11]. To order  $\mathcal{O}(\Omega_s)$ , the mode displacement vector  $\xi_\alpha e^{-i\omega t} \propto e^{im\phi - i\omega t}$  satisfies the equation

$$\xi_\alpha + iq(\mathbf{e}_z \times \xi_\alpha) = \nabla\psi, \quad (18)$$

where  $\mathbf{e}_z$  is the unit vector along the  $z$ -axis ( $\Omega_s$ ),  $q = 2\Omega_s/\omega$ , and  $\omega^2\psi = \delta P/\rho$  is the (Eulerian) enthalpy perturbation. For incompressible fluid,  $\nabla \cdot \xi_\alpha = 0$ , we have

$$\nabla^2\psi - q^2 \frac{\partial^2\psi}{\partial z^2} = 0. \quad (19)$$

This equation can be solved in an ellipsoidal coordinates  $(x_1, x_2)$ , giving  $\psi \propto P_j^m(x_1)P_j^m(x_2)$ . For a given set of  $(m, j)$ , there are  $(j-m)$  eigenvalues  $\omega_\alpha/\Omega_s$ . The modes with even  $(j-m)$  have even parity with respect to the equator, while those with odd  $(j-m)$  have odd parity. In this notation, “pure”  $r$ -modes correspond to those with  $j-m=1$  and have frequencies  $\omega_\alpha = -2\Omega_s/(1+m)$ . Such characterization of inertial modes can be generalized to stellar models which are compressible.

#### A. Even-Parity Modes

The  $m=2$ , even-parity ( $j=4, 6, \dots$ ) inertial modes can be excited by the  $l=m'=2$  tidal potential. The

frequencies of some modes are listed in Table I. Consider the two  $m=2$ ,  $j=4$  inertial modes. For uniform stellar models, both have eigenfunctions

$$\psi \propto \varpi^2 [42\mu^2 z^2 + 7(1-\mu^2)\varpi^2 - 6], \quad (20)$$

where  $\mu = q^{-1} = \omega/(2\Omega_s)$ , and  $(\varpi, \phi, z)$  are cylindrical coordinates. A direct calculation shows that

$$Q_{\alpha,22} = \int d^3x \rho \xi_\alpha^* \cdot \nabla(r^2 Y_{22}) = \int d^3x \rho \frac{2}{1+q} \left( \varpi \frac{\partial\psi}{\partial\varpi} + 2\psi \right) = 0. \quad (21)$$

Thus, to order  $\mathcal{O}(\Omega_s)$ , the tidal coupling coefficient vanishes. The first non-zero contribution arises in the order  $\mathcal{O}(\Omega_s^2)$ . Integration by part of equation (9) yields,

$$Q_{\alpha,22} = \int d^3x \delta\rho_\alpha^* r^2 Y_{22}(\theta, \phi) + \oint r^2 Y_{22} \rho \xi_\alpha^* \cdot d\hat{S}, \quad (22)$$

where  $\delta\rho_\alpha$  is the Eulerian density perturbation and  $\delta\rho_\alpha = (\omega_\alpha^2 \rho^2 / \Gamma_1 P) \psi$  with  $\Gamma_1$  being the adiabatic index. Formally,  $\Gamma_1 = \infty$  in an incompressible model. The first term on the right-hand-side of Eq. (22) scales as  $\Omega_s^2$ . The Lagrangian pressure perturbation vanishes at the surface, or  $\delta P + \xi \cdot \nabla P = 0$ . So the surface radial displacement  $\xi_{\alpha,r}$  satisfies  $\xi_\alpha = (\omega_\alpha^2/g)\psi(r=R) \propto \Omega_s^2$  (where  $g = GM/R^2$ ). As a result, the surface integral on the right-hand-side also comes in at the order of  $\mathcal{O}(\Omega_s^2)$ .

To accurately calculate tidal coupling to order  $\mathcal{O}(\Omega_s^2)$ , it will be necessary to compute the mode eigenfunction to order  $\mathcal{O}(\Omega_s^2)$ , including rotational distortion to the hydrostatic structure and the centrifugal force in the perturbation equation. This is beyond the scope of our paper. To estimate the tidal coupling coefficient, we consider a compressible, but uniform stellar model. The Eulerian density perturbation  $\delta\rho_\alpha$  is related to  $\psi$  by  $\delta\rho_\alpha = (\omega_\alpha^2 \rho^2 / \Gamma_1 P) \psi$ , and we fix  $\Gamma_1$  to a finite value. This  $\delta\rho_\alpha$  is then used in Eq. (22) to obtain  $Q_{\alpha,22}$ . The results are given in Table I.

For NS models with nonuniform density profile, no analytical solution for the inertial mode eigenfunction is generally possible. Wu [11] showed that if the star has a power-law density profile,  $\rho \propto (R^2 - r^2)^\beta$ , equation (19) is separable in the ellipsoidal coordinates and one can easily obtain eigenfunctions that are accurate to  $\mathcal{O}(\Omega_s)$ . However, the Eulerian density perturbation caused by inertial modes remains small [ $\delta\rho_\alpha/\rho = (\omega_\alpha^2/c_s^2)\psi \sim \omega_\alpha^2/(GM/R^3)\nabla^2\psi \ll \nabla \cdot \xi_\alpha$ ], we expect that  $Q_{\alpha,22}$  is of order  $\Omega_s^2$ , similar to that for the uniform density model ( $\beta=0$ ). Table I gives the mode eigenfrequencies and tidal coupling coefficients calculated using the eigenfunctions from [11] and setting  $\Gamma_1 = d \ln P / d \ln \rho$ . Corrections of order unity may arise when one is able to obtain eigenfunctions accurate to  $\mathcal{O}(\Omega_s^2)$ .

TABLE I: Frequencies and Tidal Coupling Coefficients for  $m = 2$  Inertial Modes in Various Power-law Density Models

$j$	$\beta = 0^a$		$\beta = 0.5$		$\beta = 1.0$	
	$\omega/\Omega_s^b$	$\bar{Q}^c$	$\omega/\Omega_s$	$\bar{Q}$	$\omega/\Omega_s$	$\bar{Q}$
4	-1.2319	0.084	-1.1607	0.0027	-1.1224	0.013
	0.2319	0.031	0.3830	0.0031	0.4860	0.019
6	-1.6434	0.018	-1.5820	0.0023	-1.5415	0.0022
	-0.8842	0.018	-0.8726	0.0077	-0.8671	0.0059
	0.1018	0.006	0.1780	0.0036	0.2364	0.0031
	1.0926	0.017	1.1814	0.021	1.2408	0.018

<sup>a</sup>The stellar density profile is  $\rho \propto (R^2 - r^2)^\beta$ .

<sup>b</sup> $\omega$  is the mode frequency in the rotating frame, as in  $\xi \propto e^{im\phi - i\omega t}$ .

<sup>c</sup> $Q_{\alpha,22} = \int d^3x \delta\rho_\alpha^* (r^2 Y_{22}) = \bar{Q} \hat{\Omega}_s^2$ , with  $\hat{\Omega}_s = \Omega_s (R^3/GM)^{1/2}$ .  $Q_{\alpha,22}$  is calculated with normalization  $\int d^3x \rho \xi \cdot \xi = 1$  and  $M = R = 1$ . For the uniform-density model,  $\bar{Q}$  is calculated fixing  $\Gamma_1 = 1$  (otherwise the result is zero), while for other models we use  $\Gamma_1 = d \ln P / d \ln \rho$ .

### B. Odd-Parity Modes

For an inclined orbit ( $\Theta \neq 0$ ), the  $m = 1$ , odd-parity ( $j = 2, 4, \dots$ ) modes can also be excited by the  $l = m' = 2$  tidal potential (see §IV). The relevant tidal coupling coefficient is

$$Q = Q_{\alpha,21} = \int d^3x \delta\rho_\alpha^* r^2 Y_{21}, \quad (23)$$

with  $Y_{21} = -(15/8\pi)^{1/2} \sin\theta \cos\theta e^{i\phi}$ .

The  $m = 1$ ,  $j = 2$  mode is a r-mode. For a uniform, incompressible star, the mode frequency is given by [18, 19], to order  $\mathcal{O}(\Omega_s^3)$ ,

$$\omega_\alpha/\Omega_s = -1 - 3\hat{\Omega}_s^2/4, \quad \sigma_\alpha/\Omega_s = -3\hat{\Omega}_s^2/4. \quad (24)$$

The tidal coupling strength is [6]

$$Q_{\alpha,21} = \left(\frac{3}{8\pi}\right)^{1/2} \hat{\Omega}_s^2. \quad (25)$$

For higher order ( $m = 1$ ,  $j = 4, 6, \dots$ ) modes, eigenfunctions including  $\Omega_s^2$  corrections are not available, and our results for  $Q_{\alpha,21}$  are estimates using  $\delta\rho_\alpha = (\omega_\alpha^2/\Gamma_1 P)\psi$ , with a  $\psi$  that is accurate to  $\mathcal{O}(\Omega_s)$ . The mode frequencies and coupling coefficients are listed in Table II for the  $j = 4, 6$  modes.

### IV. EFFECT OF INERTIAL MODE RESONANCE

The result of §III (see Table I) shows that the coupling coefficient of an inertial mode to the  $l = m = 2$  tidal potential has the form  $Q_{\alpha,22} = \bar{Q} \hat{\Omega}_s^2$ , with  $\bar{Q} \lesssim 0.1$ , and  $\hat{\Omega}_s = \Omega_s (R^3/GM)^{1/2}$ . For most modes,  $\bar{Q}$  is likely

 TABLE II: Same as Table I but for  $m = 1$  inertial modes

$j$	$\beta = 0$		$\beta = 0.5$		$\beta = 1.0$	
	$\omega/\Omega_s$	$\bar{Q}^a$	$\omega/\Omega_s$	$\bar{Q}$	$\omega/\Omega_s$	$\bar{Q}$
4	-1.7080	0.023	-1.6620	0.001	-1.6328	0.005
	-0.6120	0.019	-0.6511	0.009	-0.6751	0.032
	0.8200	0.140	0.9242	0.012	0.9897	0.052
6	-1.8617	0.005	-1.8308	0.001	-1.8088	0.001
	-1.3061	0.005	-1.2941	0.006	-1.2866	0.005
	-0.4404	0.030	-0.4833	0.017	-0.4126	0.001
	0.5373	0.075	0.6175	0.022	0.6726	0.019
	1.4042	0.008	1.4431	0.016	1.4688	0.011

$$^a Q_{\alpha,21} = \int d^3x \delta\rho_\alpha^* (r^2 Y_{21}) = \bar{Q} \hat{\Omega}_s^2.$$

to be significantly smaller. Given that our result for  $\bar{Q}$  should only be considered as an order-of-magnitude estimate (since we did not include  $\Omega_s^2$  correction to the mode eigenfunction), in the following we will scale our equations using  $\bar{Q} \sim 0.1$ . Similar consideration applies to the tidal coupling of odd-parity modes (Table II).

An  $m = 2$  inertial mode is resonantly excited by the  $m' = 2$  tide when  $\sigma_\alpha = \omega_\alpha + 2\Omega_s = 2\Omega_{\text{orb}}$ . We only need to keep the  $l = 2$  tidal potential, with  $W_{22} = (3\pi/10)^{1/2}$ ,  $|\mathcal{D}_{22}^{(2)}| = \cos^4(\Theta/2)$ . Thus the GW phase shift due to the resonance is

$$\Delta\Phi_{\text{GW}} = -2.55 \frac{R_{10}^5}{M_{1.4}^5 q(1+q)} \left(\frac{\Omega_s^2}{\varepsilon_\alpha \sigma_\alpha}\right) \left(\frac{\bar{Q}}{0.1}\right)^2 \hat{\Omega}_s^2 \times \left(\cos \frac{\Theta}{2}\right)^8 \text{ radian} \quad (26)$$

where  $M_{1.4} = M/(1.4M_\odot)$ ,  $R_{10} = R/(10 \text{ km})$ . Note that  $\Omega_s/(\varepsilon_\alpha \omega_\alpha) \sim 1$ , and  $\hat{\Omega}_s = (\nu_s/2170 \text{ Hz})(R_{10}^3/M_{1.4})^{1/2}$  (where  $\nu_s$  is the spin frequency). Thus, for  $\nu_s \lesssim 300 \text{ Hz}$ ,  $\Delta\Phi_{\text{GW}} \lesssim 0.05$  radian (for  $R \sim 10 \text{ km}$ ) — such a phase shift is too small to affect GW detection.

Compared to the  $m = 2$ ,  $j = 3$  r-mode (often called  $l = m = 2$  r-mode) resonance studied in Ref. [6], the phase shift associated with the inertial mode resonance (Eq. 26) is larger by a factor of  $(a/R)^2 \propto \Omega_s^{4/3}$ , because the r-mode can only be excited by the  $m = 2, l = 3$  (Newtonian) tidal potential.

For an inclined orbit ( $\Theta \neq 0$ ), the  $l = m' = 2$  tidal potential  $\propto Y_{22}(\theta_L, \phi_L) e^{-i2\Phi}$  has a component proportional to  $Y_{21}(\theta, \phi)$ , and thus can excite  $m = 1$ , odd-parity modes at the resonance  $|\sigma_\alpha| = 2\Omega_{\text{orb}}$ . For  $\sigma_\alpha > 0$  (i.e., the mode is prograde with respect to the spin in the inertial frame), from Eq. (16) with  $m = 1$  and  $m' = 2$ , and  $\mathcal{D}_{12}^{(2)} = 2\cos^3(\Theta/2)\sin(\Theta/2)$ , we find that the GW phase shift associated with the resonance is

$$\Delta\Phi_{\text{GW}} = -10.21 \frac{R_{10}^5}{M_{1.4}^5 q(1+q)} \left(\frac{\Omega_s^2}{\varepsilon_\alpha \sigma_\alpha}\right) \left(\frac{\bar{Q}}{0.1}\right)^2 \hat{\Omega}_s^2 \times \left(\cos \frac{\Theta}{2}\right)^6 \left(\sin \frac{\Theta}{2}\right)^2 \text{ radian}. \quad (27)$$

For modes with  $|\sigma_\alpha| \sim \Omega_s$  (such as the  $m = 1$ ,  $j = 4$  modes), Eq. (27) has the same behavior as Eq. (26) for the excitation of even-parity modes. The exception is the  $m = 1$ ,  $j = 2$  r-mode, which has  $\sigma_\alpha/\Omega_s = -(3/4)\hat{\Omega}_s^2$ . Since this mode is retrograde with respect to spin in the inertial frame, we use Eq. (17) to find

$$\Delta\Phi_{\text{GW}} = 162.5 \frac{R_{10}^5}{M_{1.4}^5 q(1+q)} \left( \frac{\Omega_s}{\varepsilon_\alpha} \right) \times \left( \sin \frac{\Theta}{2} \right)^6 \left( \cos \frac{\Theta}{2} \right)^2 \text{ radian.} \quad (28)$$

Thus the phase shift is always very significant for  $\Theta$  not too close to 0. However, the resonant condition  $2\Omega_{\text{orb}} = |\sigma_\alpha|$  implies that the GW frequency at the resonance is

$$\nu_{\text{GW}} = \frac{3}{4} \hat{\Omega}_s^2 \nu_s = 10.2 \frac{R_{10}^3}{M_{1.4}} \left( \frac{\nu_s}{400 \text{ Hz}} \right)^3 \text{ Hz.} \quad (29)$$

Thus, only when  $\nu_s > 400 M_{1.4}^{1/3}/R_{10}$  Hz can the resonance occur in the LIGO band.

## V. DISCUSSION

The result in this paper indicates that for neutron stars with spin frequencies  $\nu_s \lesssim 100$  Hz, resonant excitations of inertial modes during binary inspiral produce negligible phase shift  $\Delta\Phi_{\text{GW}}$  in the gravitational waveform. At such spin frequencies, only the g-mode resonance [6] and the gravitomagnetic resonance of the  $m = 2$  r-mode [7] may produce  $\Delta\Phi_{\text{GW}}$  of order 0.1 radian — such a phase shift probably does not affect the matched filtering method for detecting gravitational waves. Of course, one should keep in mind that the above number is for “canonical” neutron stars with  $M = 1.4M_\odot$ ,  $R = 10$  km. The phase shift depends strongly on the neutron star parameters: For the g-mode resonance,  $\Delta\Phi_{\text{GW}} \propto R^{3.5} M^{-4.5} \nu_{\text{GW}}^{-1}$  (where  $\nu_{\text{GW}}$  is the gravitational wave frequency at resonance, and it depends nonlinearly on the spin frequency); for the gravitomagnetic

r-mode resonance,  $\Delta\Phi_{\text{GW}} \propto R^4 M^{-10/3} \nu_{\text{GW}}^{2/3}$  (where  $\nu_{\text{GW}}$  of the same order as the spin frequency). Thus the phase shift is larger for neutron stars with larger radii.

For higher spin frequencies ( $\nu_s \gtrsim$  a few  $\times 100$  Hz), various resonances becomes important, including the f-mode resonances studied in [6] and the inertial mode resonances studied in this paper. While most of these resonances occur at the high-frequency band of the gravitational waves ( $\nu_{\text{GW}}$  is of order the spin frequency, and thus  $\gtrsim$  a few  $\times 100$  Hz), it is of interest to note that the excitation of the  $m = 1$  r-mode (the  $m = 1$ ,  $j = 2$  mode in our convention; see §III.B) occurs at the low-frequency band (tens of Hertz) of LIGO (see Eq. [29]) and produces very large phase shift (see Eq. [28]). While neutron stars with  $\nu_s \gtrsim$  a few  $\times 100$  Hz are found within binaries with a low-mass companion (e.g., white dwarfs), it is not clear that they are produced in compact binaries with another neutron star or black hole companion. If such systems do exist, then the detection the  $m = 1$  r-mode resonance should be straightforward, and would lead to a clean constraint on neutron star parameters (radius and spin).

Finally, it is worth mentioning some caveats of the present study. Our calculations of the tidal coupling coefficients for most inertial modes are approximate, since we did not take into account of the  $\mathcal{O}(\Omega_s^2)$  correction to the mode eigenfunction. More importantly, we have neglected buoyancy in the stellar models. The Brunt-Väisälä frequency inside a cold neutron star could be as large as 100 Hz [5, 20]. While the finite buoyancy has a negligible effect on the r-mode, it could affect the other inertial modes in an appreciable way. It is worthwhile to study the property (particularly the tidal coupling strength) of the inertial modes in nonisentropic neutron star models (see [21] and references therein).

## Acknowledgments

This work was supported in part by NSF grant AST 0307252 (DL) and the Natural Sciences & Engineering Council of Canada (YW).

- 
- [1] C. Cutler and K.S. Thorne, in *Proceedings of General Relativity and Gravitation XVI*, eds. N.T. Bishop and S.D. Maharaj (Singapore, World Scientific) (2002) (gr-qc/0204090)
  - [2] C. Cutler et al. Phys. Rev. Lett. **70**, 2984 (1993)
  - [3] For analytic expressions of the orbital phase error induced by quasi-equilibrium tides, see D. Lai et al. Astrophys. J. **420**, 811 (1994).
  - [4] A. Reisenegger and P. Goldreich, Astrophys. J. **426**, 688 (1994)
  - [5] D. Lai, Mon. Not. Roy. Astron. Soc. **270**, 611 (1994)
  - [6] W.C.G. Ho and D. Lai, Mon. Not. Roy. Astron. Soc. **308**, 153 (1999).
  - [7] E.E. Flanagan and E. Racine, Phys. Rev. D. submitted (2006).
  - [8] J. Papaloizou and J.E. Pringle, MNRAS, **195**, 743 (1981)
  - [9] K.H. Lockitch and J.L. Friedman, Astrophys. J. **521**, 764 (1999)
  - [10] A.K. Schenk, et al. Phys. Rev. **D 65**, 024001 (2002)
  - [11] Y. Wu, Astrophys. J. **635**, 674 (2005)
  - [12] W.H. Press and S.A. Teukolsky, Astrophys. J. **213**, 183 (1977)
  - [13] J.L. Friedman and B.F. Schutz, Astrophys. J. **221**, 937 (1978)
  - [14] J. Dyson and B.F. Schutz, Proc. Royal Soc. London A **368**, 389 (1979)
  - [15] In Ref.[6], mode decomposition was carried out in the configuration space (i.e.,  $\xi = \sum_\alpha c_\alpha \xi_\alpha$ ), which is in

general not valid. Nevertheless, if one *assumes* that the modes are orthogonal with each other, one obtains the same expression for the energy transfer. See also [16] for a discussion of this issue.

- [16] D. Lai, *Astrophys. J.* **490**, 847 (1997)
- [17] Y. Wu, *Astrophys. J.* **635**, 688 (2005)
- [18] H. Saio, *Astrophys. J.* **256**, 717 (1982)
- [19] J.D. Kokkotas and N. Stergioulas, *Astron. & Astrophys.* **341**, 110 (1999)
- [20] A. Reisenegger and P. Goldreich, *Astrophys. J.* **395**, 240 (1992)
- [21] S. Yoshida and U. Lee, *Astrophys. J. Supp.* **129**, 353 (2000)